e content for students of patliputra university

B. Sc. (Honrs) Part 2paper 3

Subject:Mathematics

Title/Heading of topic:Sub group , intersection of sub groups, Subgroup of cycle groups

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Definition 2 - 1 .

Let $(G, *)$ be a group and $H \subseteq G$ be a nonempty subset of G . The pair $(H, *)$ is said to be a SUBGROUP of $(G, *)$ if $(H, *)$ is group .

Example .

$(Z, +)$ is subgroup of $(R, +)$.

Note .

Each group $(G, *)$ has at least two subgroups .

$(\{e\}, *)$ and $(G, *)$ are subgroups of $(G, *)$, these two subgroups are called trivial subgroups .

Exercise .

1) Find all subgroups of the group $(Z_8, +_8)$.

2) Find all subgroups of (S_3, o) and (S_4, o).

Theorem 2-1.

Let $(G, *)$ be a group and $\phi \neq H \subseteq G$. Then $(H, *)$ is a subgroup of $(G, *)$ if and only if $a, b \in H$ implies $a * b^{-1} \in H$.

Proof .

If $(H, *)$ is a subgroup and $a, b \in H$,

then $b^{-1} \in H$,

So $a * b^{-1} \in H$?

Conversely ,

$H \neq \phi$

Since $a * b^{-1} \in H$ whenever $a, b \in H$,

So we can take $a = b$.

Then $b * b^{-1} = e \in H$.

$b^{-1} = e * b^{-1} \in H$ for every b in H.

$a * b = a * (b^{-1})^{-1} \in H$?

hence H is closed.

If $a, b, c \in H$, then $a, b, c \in G$.

$a * (b * c) = (a * b) * c$?

$*$ is associative on H.

Hence $(H, *)$ is subgroup.

Theorem 2 – 2.

The intersection of two subgroups of the group is subgroup.

Proof.

Suppose that $(H, *)$ and $(K, *)$ are subgroups of group $(G, *)$.

We must prove that $(H \cap K, *)$ is subgroup.

Since $(H, *)$ and $(K, *)$ are subgroups,

Then $\exists\, e \in H$ and $e \in K$

Hence $e \in H \cap K$.

So $H \cap K \neq \phi$.

If $a, b \in H \cap K$, then $a, b \in H$ and $a, b \in K$,

Hence $a * b^{-1} \in H$ and $a * b^{-1} \in K$.

So $a * b^{-1} \in H \cap K$.

Definition 2 – 2 .

Let $(G , *)$ be group , $(G , *)$ is called commutative group if and only if ,

$a * b = b * a$ for all $a , b \in G$.

Example 1.

$(R , +)$ is commutative group , for

$a + b = b + a$ for all $a , b \in R$.

Example 2.

(S_3 , o) is not commutative group .

Definition 2 – 3 .

The center of a group $(G , *)$, denoted by cent G , is the set

$\text{cent } G = \{ c \in G \mid c * x = x * c \text{ for all } x \in G \}$.

Example 1.

Find center of the group $(Z , +)$.

If $n \in Z$,

Then $n + m = m + n$ for all $m \in Z$.

So $\text{cent } Z = Z$

Example 2.

$\text{cent } S_3 = \{ e \}$.

Exercise .

Find $\text{cent } Z_8$.

Note .

For any group $(G , *)$, $\text{cent } G \neq \phi$.

Theorem 2 – 3 .

(cent G , *) is subgroup of each group (G , *) .

Proof .

cent G $\neq \phi$.

If a , b $\in$ cent G , then for every x $\in$ G ,

$a * x = x * a$ and

$b * x = x * b$.

$$(a * b^{-1}) * x = a * (b^{-1} * x)$$
$$= a * (x^{-1} * b)^{-1}$$
$$= a * (b * x^{-1})^{-1}$$
$$= a * (x * b^{-1})$$
$$= (a * x) * b^{-1}$$
$$= (x * a) * b^{-1}$$
$$= x * (a * b^{-1})$$

Hence $a * b^{-1} \in$ cent G .

So (cent G , *) is subgroup.

Exercise .

Is the union of two subgroups group ?

H . W.

Theorem 2 – 4 .

Let (H_1 ,*) and (H_2 ,*) be subgroups of the group (G ,*) . ($H_1 \cup H_2$,*) is also subgroup

and Only if $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

Proof .

Suppose that ($H_1 \cup H_2$, *) is subgroup , we must prove that $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

If $H_1 \subseteq H_2$ and $H_2 \subseteq H_1$.

Then ∃ $a \in H_1 - H_2$ and $b \in H_2 - H_1$.

If $a * b \in H_1$, then ,

$b = a^{-1} * (a * b) \in H_1$,

which is contradiction .

If $a * b \in H_2$, then ,

$a = (a * b) * b^{-1} \in H_2$,

which is contradiction.

Hence $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

Conversely , suppose that ,

$H_1 \subseteq H_2$ or $H_2 \subseteq H_1$, we must prove that ($H_1 \cup H_2$, *) is subgroup .

If $H_1 \subseteq H_2$, then $H_1 \cup H_2 = H_2$,

then ($H_1 \cup H_2$,*) is subgroup .

If $H_2 \subseteq H_1$, then $H_1 \cup H_2 = H_1$,

then ($H_1 \cup H_2$,*) is subgroup .

Example .

If (Z_{12} , $+_{12}$) is a group ,

(H_1 = { [0] , [3] ,[6] , [9] } , $+_{12}$) and (H_2 = { [0] , [6] } , $+_{12}$) are subgroups of (Z_{12} , $+_{12}$)?

Then $H_1 \cup H_2 = H_1$.

But if H_3 = { [o] , [4], [8] } .

Then ($H_1 \cup H_3$, $+_{12}$) is not subgroup ?

CYCLIC GROUP

Definition 2 – 4 .

If $(G, *)$ is an arbitrary group and $\phi \neq S \subseteq G$, then (S) will represent the set ,

$$(S) = \cap \{ H \mid S \subseteq H ; (H, *) \text{ is a subgroup of } (G, *) \}.$$

Note .

$((S), *)$ is a subgroup of $(G, *)$.

Definition 2 – 5 .

Let $(G, *)$ be a group , the subgroup $((S), *)$ is called the subgroup generated by the set S .

Example .

$(Z, +)$ is generated by $(Z_o, +)$.

Note .

If S consists of a single element a , then $((a), *)$ is called the cyclic subgroup generated by a .

$(a) = \{ a^n \mid n \in Z \}$.

Definition 2 – 6 .

A group is cyclic means that each of its members can be expressed as an integral power of some fixed element of the group .

Example 1 .

$(Z_{10}, +_{10})$ is cyclic group .

Example 2.

(S_3, o) is not cyclic group .

Definition 2 – 7 .

The ORDER of a group is the number of its elements .

Theorem 2 – 5 .

If $((a), *)$ is a finite cyclic group of order n , then

$(a) = \{ e, a, a^2, a^3, \dots, a^{n-1} \}$.

Proof .

Since (a) is finite , then not all powers of a are distinet .

$\exists a^i = a^j \qquad i < j$.

Then $\qquad a^i * a^{-i} = a^j * a^{-i}$.

$a^{j-i} = e$.

Thus the set of positive integers k for which $a^k = e$ is nonempty .

Suppose that m is the smallest positive integer such that $a^m = e$ and $a^k \neq e$ for $0 < k < m$

The set $\quad S = \{ e, a, a^2, \dots, a^{m-1} \}$ consiste of distinet elements of (a) .

By the division algorithm $\quad k = qm + r \quad o \leq r < m$.

Hence $\qquad a^k = (a^m)^q * a^r = e * a^r = a^r \in S$.

Then $\qquad (a) \subseteq S$.

Hence $\qquad (a) = S , \quad m = n$.

Theorem 2 – 6.

Every subgroup of a cyclic group is cyclic .

Proof .

Let $((a), *)$ be a cyclic group generated by a and $(H, *) \subseteq ((a), *)$.

If $H = \{e\}$, then H is cyclic ?

If $H = (a)$, then H is cyclic ?

If H is proper set and $a^m \in H$ where $m \neq 0$,

Then $a^{-m} \in H$?

Hence , H must contain positive powers of a .

Let n be the smallest positive integer $\ni a^n \in H$.

We must prove that $H = (a^n)$.

Let $a^k \in H$.

From division algorithm ,$\exists$ integers q and r $\ni \quad K = qn + r \qquad 0 \leq r < n$.

If $r > 0$, then contradiction ?

then $r = o$ and $k = qn$.

Hence only powers of $a^n \in H$, then $H \subseteq (a^n)$.

Since H is closed , then any power of a^n must be in H .

Hence $(a^n) \subseteq H$.

Then $H = (a^n)$.

Definition 2-8.

Let (G,*) be a group and H , K be nonempty subsets of G. The product of H and K, in that order , is the set

$$H*K = \{ h*k \mid h \in H , k \in K \}$$

Note .

(H* K , *) is not group always .

Example .

If $H = \{ R_{360} , D_1 \}$ and $K = \{ R_{360} , V \}$,

Then

$$H * K = \{ R_{360} * R_{360} , R_{360} * V , D_1 * R_{360} , D_1 * V \}$$

$$= \{ R_{360} , V , D_1 , R_{270} \}$$

It is clear that H* K not group ?

Theorem 2 – 7 .

If (H , *) and (K , *) are subgroups of the group (G , *) such that $H * K = K * H$, then (H*K ,*) is also a subgroup .

Proof .

$H * K \neq \phi$?

Let $a , b \in H * K$, then $a = h*k$ and $b = h_1 * k_1$, for suitable choice of $h , h_1 \in H$ and $k , k_1 \in K$

$$a * b^{-1} = (h *k) * (h_1 * k_1)^{-1}$$

$$= (h * k)* (k_1^{-1} * h_1^{-1})$$

$$= h * ((k * k_1^{-1}) * h_1^{-1})$$

Since K is subgroup , then $k * k_1^{-1} \in K$ and $(k * k_1^{-1}) * h_1^{-1} \in K * H$

Hence $(k * k_1^{-1}) * h_1^{-1} \in H * K$?

So $\exists\ h_2 \in H$ and $k_2 \in K \ni$

$(k * k_1^{-1}) * h_1^{-1} = h_2 * k_2,$

Then $a * b^{-1} = h * (h_2 * k_2) = (h * h_2) * k_2 \in H * K$?

Corollary 1.

If $(H, *)$ and $(K, *)$ are subgroups of the commutative group $(G, *)$ then $(H*K, *)$ is again a subgroup.

Corollary 2.

If $(H*K, *)$ is a subgroup of $(G, *)$, then $(H * K, *) = ((H \cup K), *)$.

Exercises